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CALCULATION OF LOSSES FROM A DIELECTRIC ROD.(U)
JUL 81 J P COUGHLIN
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CALCULATION OF LOSSES FROM A DIELECTRIC ROD

BY JAMES P. COUGHLIN

RESEARCH AND TECHNOLOGY DEPARTMENT

JULY 1981

NAVAL SURFACE WEAPONS CENTER

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Power loss computations are made for the dielectric rod at a step or over a gradual taper.

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FOREWORD

Computer programs were written to calculate the power losses of the dominant mode on a dielectric rod at points where the rod has a change in diameter due to either a sharp step or a gradual taper. The equations used for the computation are those of Dietrich Marcuse. The first program calculates the power losses at a step in terms of the step size and the dielectric constant. The second calculates the power losses for tapered rods of differing lengths, dielectric constants and taper shapes. The results are summarized in the tables.

Ira M. Blatstein

IRA M. BLATSTEIN

By direction

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PART A: GENERAL THEORY

The equations governing the power losses of a cylindrical dielectric rod due to variations in radius and dielectric constants have been deduced by D. Marcuse.[1] Starting from the solutions to the Maxwell equations, he derives the expression for the power loss in terms of the coefficients of transmission and reflection at a "step" (VIZ: a jump discontinuity in the radius). These coefficients are calculated using the approximation that the reflected radiation modes are approximately orthogonal to the incident modes.

Assuming cylindrical symmetry, the fields of the incident bound mode are:

$$E_Z = A\, J_\nu (\kappa r) \cos \nu \phi \tag{1a}$$

$$H_Z = B\, J_\nu (\kappa r) \sin \nu \phi \tag{1b}$$

$$E_r = \frac{-i}{\kappa^2} \left[\kappa \beta_o A J'_\nu (\kappa r) + \omega \mu B \frac{\nu}{r} J_\nu (\kappa r) \right] \cos \nu \phi \tag{1c}$$

$$E_\phi = \frac{i}{\kappa^2} \left[\beta_o A \frac{\nu}{r} J_\nu (\kappa r) + \kappa \omega \mu B J'_\nu (\kappa r) \right] \sin \nu \phi \tag{1d}$$

[1] Marcuse, Dietrich, "Radiation Losses of the Dominant Mode in Round Dielectric Waveguides," Bell System Technical Journal, Oct 1970, p. 1665.ff.

$$H_r = -\frac{i}{\kappa^2}\left[n^2\,\omega\,\varepsilon_o\,A\,\frac{\nu}{r}\,J_\nu\,(\kappa r) + \kappa\,\beta_o\,B\,J_\nu'(\kappa r)\right]\sin\nu\,\phi \tag{1e}$$

$$H_\phi = -\frac{i}{\kappa^2}\left[n^2\,\kappa\,\omega\,\varepsilon_o\,A\,J_\nu'\,(\kappa r) + \beta_o\,B\,\frac{\nu}{r}\,J_\nu\,(\kappa r)\right]\cos\nu\,\phi \tag{1f}$$

Where ν is an integer, J_ν is the ν^{th} order Bessel function and the factor $\exp i(\omega t - \beta_o z)$ has been omitted from each term.

$$k^2 = \omega^2\,\varepsilon_o\,\mu_o \tag{2a}$$

$$\kappa^2 = n^2\,k^2 - \beta_o^2 \tag{2b}$$

n^2 = dielectric constant

β_o = propagation constant to be determined by the boundary conditions.

The fields outside the cylindrical rod are:

$$E_Z = C\,H_\nu^1\,(i\gamma r)\cos\nu\,\phi \tag{3a}$$

$$H_Z = D\,H_\nu^1\,(i\gamma r)\sin\nu\,\phi \tag{3b}$$

$$E_r = \frac{i}{\gamma^2}\left[i\,\gamma\,\beta_o\,C\,H_\nu^{(1)'}(i\gamma r) + \omega\,\mu\,D\,\frac{\nu}{r}\,H_\nu^{(1)}\,(i\gamma r)\right]\cos\nu\,\phi \tag{3c}$$

$$E_\phi = -\frac{i}{\gamma^2}\left[\beta_o\,C\,\frac{\nu}{r}\,H_\nu^{(1)}\,(i\gamma r) + i\,\omega\,\mu\,\gamma\,D\,H_\nu^{(1)'}\,(i\gamma r)\right]\sin\nu\,\phi \tag{3d}$$

$$H_r = \frac{i}{\gamma^2}\left[\omega\,\varepsilon_o\,C\,\frac{\nu}{r}\,H_\nu^{(1)}\,(i\gamma r) + i\,\gamma\,\beta_o\,D\,H_\nu^{(1)'}\,(i\gamma r)\right]\sin\nu\,\phi \tag{3e}$$

$$H_\phi = \frac{i}{\gamma^2}\left[i\,\gamma\,\omega\,\varepsilon_o\,C\,H_\nu^{(1)'}\,(i\gamma r) + \beta_o\,D\,\frac{\nu}{r}\,H_\nu^{(1)}\,(i\gamma r)\right]\cos\nu\,\phi \tag{3f}$$

Where the H's are the Hankel functions of the first kind of order ν and

$$\gamma^2 = \beta_o^2 - k^2 \tag{4}$$

To insure the continuity of the fields at the interface, the following eigenvalue equation must be satisfied by β_o:

$$\left[n^2 \frac{a\gamma^2}{\kappa}\left(\frac{J_o\ (\kappa a)}{J_1\ (\kappa a)} - \frac{1}{\kappa a}\right) + \gamma a \frac{i\ H_o^{(1)}\ (i\gamma a)}{H_1^{(1)}\ (i\gamma a)} - 1\right]$$

$$\left[\frac{a\gamma^2}{\kappa}\left(\frac{J_o\ (\kappa a)}{J_1\ (\kappa a)} - \frac{1}{\kappa a}\right) + \gamma a \frac{i\ H_o^{(1)}\ (i\gamma a)}{H_1^{(1)}\ (i\gamma a)} - 1\right]$$

$$= \left[(n^2 - 1)\frac{\beta_o k}{\kappa^2}\right]^2 \tag{5}$$

(We have set $\nu = 1$, specializing to the lowest order mode.) When this eigenvalue equation is satisfied, the condition of continuity determines all the coefficients in terms of A. The coefficient, A, is determined by the power in the mode, as is discussed below.

The fields described so far are the bound modes and they fall off rapidly with distance from rod axis. Inside the dielectric, the expressions for the fields of the radiation modes are the same as for the bound modes with A, B and κ replaced by F, G and σ respectively. Outside the rod, these fields are given by:

$$E_Z = \left[H\, J_\nu(\rho r) + I\, Y_\nu(\rho r)\right] \cos \nu\phi \tag{6a}$$

$$H_Z = \left[K\, J_\nu(\rho r) + M\, Y_\nu(\rho r)\right] \sin \nu\phi \tag{6b}$$

$$E_r = -\frac{i}{\rho^2}\left\{\rho\,\beta\left[H\, J'_\nu(\rho r) + I\, Y'_\nu(\rho r)\right]\right.$$

$$\left. + \omega\,\mu\,\frac{\nu}{r}\left[K\, J_\nu(\rho r) + M\, Y_\nu(\rho r)\right]\right\} \cos \nu\phi \tag{6c}$$

$$E_\phi = \frac{i}{\rho^2}\left\{\beta\,\frac{\nu}{r}\left[H\, J_\nu(\rho r) + I\, Y_\nu(\rho r)\right]\right.$$

$$\left. + \rho\,\omega\,\mu\left[K\, J'_\nu(\rho r) + M\, Y'_\nu(\rho r)\right]\right\} \sin \nu\phi \tag{6d}$$

$$H_r = -\frac{i}{\rho^2}\left\{\omega\,\varepsilon_o\,\frac{\nu}{r}\left[H\, J_\nu(\rho r) + I\, Y_\nu(\rho r)\right]\right.$$

$$\left. + \rho\,\beta\left[K\, J'_\nu(\rho r) + M\, Y'_\nu(\rho r)\right]\right\} \sin \nu\phi \tag{6e}$$

$$H_\phi = -\frac{i}{\rho^2}\left\{\rho\,\omega\,\varepsilon_o\left[H\, J'_\nu(\rho r) + I\, Y'_\nu(\rho r)\right]\right.$$

$$\left. + \beta\,\frac{\nu}{r}\left[K\, J_\nu(\rho r) + M\, Y_\nu(\rho r)\right]\right\} \cos \nu\phi \tag{6f}$$

where $\rho^2 = k^2 - \beta^2$ (7)

The extra coefficients arise from the fact that we can no longer require the fields to approach zero so rapidly. Their presence makes it possible to insure the continuity of the fields without the eigenvalve equation being satisfied. The equations arising from these conditions are sufficient to determine the coefficients H, I, K and M used above, and also to determine the ratio of the coefficients (We specialize to the case $\nu = 1$):

$$\frac{F}{G} = \sqrt{\frac{\mu_o}{\varepsilon_o}} \left(\frac{(g-b)^2 + (e-d)^2 + c^2 + f^2}{(g-n^2 b)^2 + (e-n^2 d)^2 + c^2 + f^2} \right)^{1/2} \tag{8}$$

where $b = \rho/\sigma \; J_1'(\sigma a) \; Y_1(\rho a)$ (9a)

$$c = \frac{(n^2 - 1)\, k\, \beta}{\rho\, \sigma^2\, a} J_1(\sigma a)\, Y_1(\rho a) \tag{9b}$$

$$d = \rho/\sigma \; J_1'(\sigma a)\; J_1(\rho a) \tag{9c}$$

$$e = J_1(\sigma a)\; J_1'(\rho a) \tag{9d}$$

$$f = \frac{(n^2 - 1)\, k\, \beta}{\rho\, \sigma^2\, a} J_1(\sigma a)\, J_1(\rho a) \tag{9e}$$

$$g = J_1(\sigma a)\; Y_1'(\rho a) \tag{9f}$$

$$\sigma^2 = n^2 k^2 - \beta^2 \tag{10}$$

This equation has two roots corresponding to the different choices of sign for the ratio F/G. As Dr. Marcuse indicates,[1] this corresponds to the even and odd modes to be found in slab-symmetric problems (cos ϕ and sin ϕ terms). To completely specify the problem, we have to specify the power in each mode:

$$P = \left(\frac{\pi}{2}\right)^3 \frac{a^2 \beta}{\rho}\, \omega\, \varepsilon_o \left\{ \left[g - n^2 b + c \sqrt{\frac{\mu_o}{\varepsilon_o}} \frac{G}{F} \right]^2 + \left[e - n^2 d + f \sqrt{\frac{\mu_o}{\varepsilon_o}} \frac{G}{F} \right]^2 + \left[c + (g-b) \sqrt{\frac{\mu_o}{\varepsilon_o}} \frac{G}{F} \right]^2 + \left[f + (e-d) \sqrt{\frac{\mu_o}{\varepsilon_o}} \frac{G}{F} \right]^2 \right\} F^2 \tag{11}$$

[1] See footnote 1 on page 7.

where b, c, d, e, f and g are as above.

When the power in the mode is specified: then, for each choice of sign of F/G, F can be calculated and then G and finally H, I, K and M giving a full description of the incident fields.

When the incident fields strike a step, the continuity of the tangential components of the electric and magnetic fields requires that

$$E_r^{(i)} + a_r E_r^{(r)} + \int_0^{\infty} \left[q_r(\rho)\, E_{r,1}^{(r)}(\rho) + p_r(\rho)\, E_{r,2}^{(r)}(\rho) \right] d\rho$$

$$= c_t E_r^{(t)} + \int_0^{\infty} \left[q_t(\rho)\, E_{r,1}^{(t)}(\rho) + p_t(\rho)\, E_{r,2}^{(t)}(\rho) \right] d\rho \qquad (12)$$

with similar equations for E_ϕ, E_z, H_r, H_ϕ and H_z

c_t = coefficient of transmission

a_r = coefficient of reflection

q (ρ) = the amplitude corresponding to the amount of power scattered into the mode, P, for the first choice of sign for F/G

p (ρ) = same as q (ρ) with the sign of F/G reversed.

The superscripts i, t and r refer to the incident, transmitted and reflected waves respectively.

An exact solution to the resulting set of equations would be extremely difficult. However, an approximate solution to the problem can be obtained by ignoring the reflected radiation when calculating the coefficient, c_t. If the steps are large, the radiation will be predominantly in the forward direction, so neglecting the backward radiation seems justifiable in this case. If the steps are small, then the radiation modes of the reflected radiation are nearly orthogonal to the incident radiation modes and our approximation would seem justified in this case, also.

With this approximation, it is possible to calculate the transmission coefficient, c_t, and also the reflection coefficient, a_r, at a "step". If P represents the incident power, then:

$$c_t = \frac{2I_1 I_2}{(I_1 + I_2)P} \tag{13}$$

$$a_r = \frac{I_1 - I_2}{I_1 + I_2} \tag{14}$$

where the two integrals are given by:

$$
\begin{aligned}
I_1 = \frac{\pi}{2} \Bigg\{ & \frac{1}{\gamma_2^2} (\beta_1 A_1 - \omega \mu B_1)(\omega \varepsilon_o A_2 - \beta_2 B_2) \frac{J_1(\kappa_2 a_2)}{H_1^1(i \gamma_2 a_2)} * \\
& \Bigg[\left(\frac{1}{\kappa_1^2} + \frac{1}{\gamma_1^2} \right) J_1(\kappa_1 a_1) H_1^1(i \gamma_2 a_1) - \frac{1}{\kappa_1^2} J_1(\kappa_1 a_2) H_1^1(i \gamma_2 a_2) \Bigg] \\
& - \frac{1}{\kappa_1^2 \kappa_2^2} (\beta A_1 - \omega \mu B_1)(n^2 \omega \varepsilon_o A_2 - \beta_2 B_2) J_1(\kappa_1 a_2) J_1(\kappa_2 a_2) \\
& + \frac{a_2}{\kappa_1 \kappa_2 \left(\kappa_1^2 - \kappa_2^2 \right)} (\omega \varepsilon_o n^2 \beta_1 A_1 A_2 + \omega \mu \beta_2 B_1 B_2) * \\
& \Big[\kappa_1 J_1(\kappa_1 a_2) J_o(\kappa_2 a_2) - \kappa_2 J_o(\kappa_1 a_2) J_1(\kappa_2 a_2) \Big] \\
& + \frac{1}{\gamma_2} (\omega \varepsilon_o \beta_1 A_1 A_2 + \omega \mu \beta_2 B_1 B_2) \frac{J_1(\kappa_2 a_2)}{H_1^1(i \gamma_2 a_2)} * \\
& \Bigg[\frac{1}{\kappa_1^2 + \gamma_2^2} \Big[(i a_2 J_1(\kappa_1 a_2) H_o^1(i \gamma_2 a_2) - i a_1 J_1(\kappa_1 a_1) H_o^1(i \gamma_2 a_1) \Big] \\
& + \frac{\gamma_2}{\kappa_1} \Big[(a_2 J_o(\kappa_1 a_2) H_1^1(i \gamma_2 a_2) - a_1 J_o(\kappa_1 a_1) H_1^1(i \gamma_2 a_1) \Big] \\
& + \frac{a_1}{\gamma_2^2 - \gamma_1^2} \frac{J_1(\kappa_1 a_1)}{H_1^1(i \gamma_1 a_1)} \Bigg(i H_1^1(i \gamma_1 a_1) H_o^1(i \gamma_2 a_1) \\
& - i \frac{\gamma_2}{\gamma_1} H_o^1(i \gamma_1 a_1) H_1^1(i \gamma_2 a_1) \Bigg) \Bigg] \Bigg\}
\end{aligned}
\qquad (15)
$$

$$
\begin{aligned}
I_2 = \frac{\pi}{2} \Bigg\{ & -\frac{1}{\kappa_1^2} \left(n^2 \omega \varepsilon_o A_1 - \beta_1 B_1\right)\left(\beta_2 A_2 - \omega \mu B_2\right) \frac{J_1(\kappa_2 a_2)}{H_1^1(i \gamma_2 a_2)} \\
& \left[\left(\frac{1}{\kappa_2^2} + \frac{1}{\gamma_2^2}\right) J_1(\kappa_1 a_2) H_1^1(i\gamma_2 a_2) - \frac{1}{\gamma_2^2} J_1(\kappa_1 a_1) H_1^1(i \gamma_2 a_1)\right] \\
& + \frac{1}{\gamma_1^2 \gamma_2^2} \left(\omega \varepsilon_o A_1 - \beta_1 B_1\right)\left(\beta_2 A_2 - \omega \mu B_2\right) \frac{J_1(\kappa_2 a_2)}{H_1^1(i\gamma_2 a_2)} J_1(\kappa_1 a_1) H_1^1(i\gamma_2 a_1) \\
& + \frac{a_1}{\gamma_1 \gamma_2 \left(\gamma_2^2 - \gamma_1^2\right)} \left(\omega \varepsilon_o \beta_2 A_2 A_1 + \omega \mu \beta_1 B_1 B_2\right) \frac{J_1(\kappa_1 a_1)}{H_1^1(i\gamma_1 a_1)} \\
& \frac{J_1(\kappa_2 a_2)}{H_1^1(i\gamma_2 a_2)} \left[i\gamma_1 H_1^1(i\gamma_1 a_1) H_o^1(i\gamma_2 a_1) - i\gamma_2 H_0^1(i\gamma_1 a_1) H_1^1(i\gamma_2 a_1)\right] \\
& + \frac{1}{\kappa_1} \left(n^2 \omega \varepsilon_o \beta_2 A_1 A_2 + \omega \mu \beta_1 B_1 B_2\right) \\
& \Bigg[\frac{a_2}{\kappa_1^2 - \kappa_2^2} \left(\frac{\kappa_1}{\kappa_2} J_1(\kappa_1 a_2) J_o(\kappa_2 a_2) - J_o(\kappa_1 a_2) J_1(\kappa_2 a_2)\right) \\
& + \frac{1}{\kappa_1^2 + \gamma_2^2} \frac{J_1(\kappa_2 a_2)}{H_1^1(i\gamma_2 a_2)} \Bigg(a_2 J_o(\kappa_1 a_2) H_1^1(i\gamma_2 a_2) - a_1 J_o(\kappa_1 a_1) H_1^1(i\gamma_2 a_1) \\
& + \frac{\kappa_1}{\gamma_2}\left(i a_2 J_1(\kappa_1 a_2) H_o^1(i\gamma_2 a_2) - i a_1 J_1(\kappa_1 a_1) H_o^1(i\gamma_2 a_1)\right)\Bigg)\Bigg]\Bigg\}
\end{aligned}
\tag{16}
$$

The subscripts 1 and 2 indicate a quantity found to the left of the step or to the right, respectively. The quantities, A and B, are the amplitudes of the electric and magnetic fields inside the rod. P is the power associated with the dominant mode ($\nu = 1$). With the aid of these coefficients, the power loss becomes:

$$\frac{\Delta P}{P} = 1 - |c_t|^2 - |a_r|^2 \tag{17}$$

This equation is suitable for the power loss computation for an abrupt step. Alternatively, the power carried away in the radiation modes could be calculated directly from:

$$\frac{\Delta P}{P} = \int_{-k}^{k} \left\{|q|^2 + |p|^2\right\} \frac{|\beta|}{\rho} \, d\beta \tag{18}$$

Both reflected and transmitted modes are automatically included by integrating from -k to k instead of 0 to k. If we take $a = a(z)$, then $\Delta a = \frac{da}{dz} \Delta z$ and Δa will be small so that an abrupt step is replaced by a sequence of small steps. The resulting value of q is:

$$q(\rho) = \int_{o}^{L} I(\rho, z) \frac{da}{dz} \exp\left[-i \int_{o}^{z} (\beta_o - \beta) \, ds\right] dz \tag{19}$$

The function q (which governs the losses of the "even" modes) corresponds to one choice of sign for F/G, while p (ρ) corresponds to the other choice (see the expression for F/G above).

The function, $I(\rho, z)$, is calculated by expressing quantities to the right of a small step in terms of quantities on the left and a Taylor series expansion:

$$F(a_2) = F(a_1) + \left(\frac{\partial F}{\partial a}\right)_{a=a_1} \Delta a + \ldots$$

the higher order terms being neglected.

$$I(\rho, z) = \frac{\pi}{4\rho^2 \gamma^2 P} J_1(\kappa a) *$$

$$\left\{(\beta_o + \beta)\gamma\rho\left(\omega\,\varepsilon_o\,A\frac{\partial H}{\partial a} + \omega\,\mu\,B\frac{\partial K}{\partial a}\right) *\right.$$

$$\left[a\,\frac{\gamma J_o(\rho a) + i\rho \dfrac{H_o^1(i\gamma a)}{H_1^1(i\gamma a)} J_1(\rho a)}{\gamma^2 + \rho^2} - \frac{1}{\gamma\rho} J_1(\rho a)\right]$$

$$+ (\beta_o + \beta)\gamma\rho\left(\omega\,\varepsilon_o\,A\frac{\partial I}{\partial a} + \omega\,\mu\,B\frac{\partial M}{\partial a}\right)*$$

$$\left[a \frac{\gamma Y_o(\rho a) + i\rho \dfrac{H_o^1(i\gamma a)}{H_1^1(i\gamma a)} Y_1(\rho a)}{\gamma^2 + \rho^2} - \frac{1}{\gamma\rho} Y_1(\rho a) \right]$$

$$+ (k^2 + \beta_o \beta) \left[\left(A \frac{\partial K}{\partial a} + B \frac{\partial H}{\partial a} \right) J_1(\rho a) \right.$$

$$\left. + \left(A \frac{\partial M}{\partial a} + B \frac{\partial I}{\partial a} \right) Y_1(\rho a) \right] \Bigg\} \tag{20}$$

The partial derivatives required for this function are:

$$\frac{\partial H}{\partial a} = \frac{\pi\rho}{2} \left[\left\{ a \frac{\sigma^2 - n^2\rho^2}{\sigma} J_o(\sigma a) Y_1'(\rho a) + \left[\frac{2}{\rho a} - \rho a + n^2 \left(\rho a - \frac{2\rho}{a\sigma^2} \right) \right] \right. \right.$$

$$\left. J_1(\sigma a) Y_1(\rho a) + (n^2 \rho^2/\sigma^2 - 1) J_1(\sigma a) Y_o(\rho a) \right\} F$$

$$+ \frac{(n^2 - 1) k^2 \beta}{\omega \varepsilon_o \sigma^2 \rho} \Big\{ \sigma J_o(\sigma a) Y_1(\rho a) + \rho J_1(\sigma a) Y_o(\rho a)$$

$$\left. - \frac{2}{a} J_1(\sigma a) Y_1(\rho a) \Big\} G \right] \tag{21a}$$

$$\frac{\partial I}{\partial a} = -\frac{\pi\rho}{2} \left[\left\{ a \frac{\sigma^2 - n^2\rho^2}{\sigma} J_o(\sigma a) J_1'(\rho a) + \left[\frac{2}{\rho a} - \rho a + n^2 \left(\rho a - \frac{2\rho}{a\sigma^2} \right) \right] \right. \right.$$

$$\left. J_1(\sigma a) J_1(\rho a) + (n^2 \rho^2/\sigma^2 - 1) J_1(\sigma a) J_o(\rho a) \right\} F$$

$$+ \frac{(n^2 - 1)\, k^2\, \beta}{\omega\, \varepsilon_o\, \rho\sigma^2} \left\{ \sigma\, J_o(\sigma a)\, J_1(\rho a) + \rho\, J_1(\sigma a)\, J_o(\rho a) \right.$$

$$\left. - \frac{2}{a} J_i(\sigma a)\, J_1(\rho a) \right\} G \Bigg] \tag{21b}$$

$$\frac{\partial K}{\partial a} = \frac{\pi\, \rho}{2\sigma} (n^2 - 1)\, k^2 \left[\frac{\beta}{\omega\, \mu\, \rho\, \sigma} \left\{ \sigma\, J_o(\sigma a)\, Y_1(\rho a) \right. \right.$$

$$\left. + \rho\, J_1(\sigma a)\, Y_o(\rho a) - \frac{2}{a} J_1(\sigma a)\, Y_1(\rho a) \right\} F + \left\{ a\, J_o(\sigma a)\, Y_1'(\rho a) \right.$$

$$\left. \left. + \frac{2}{\rho\, \sigma\, a} J_1(\sigma a)\, Y_1(\rho a) - \frac{1}{\sigma} J_1(\sigma a)\, Y_o(\rho a) \right\} G \right] \tag{21c}$$

$$\frac{\partial M}{\partial a} = - \frac{\pi\, \rho}{2\sigma} (n^2 - 1)\, k^2 \left[\frac{\beta}{\omega\, \mu\, \rho\, \sigma} \left\{ \sigma\, J_o(\sigma a)\, J_1(\rho a) \right. \right.$$

$$\left. + \rho\, J_1(\sigma a)\, J_o(\rho a) - \frac{2}{a} J_1(\sigma a)\, J_1(\rho a) \right\} F + \left\{ a\, J_o(\sigma a)\, J_1'(\rho a) \right.$$

$$\left. \left. + \frac{2}{\rho\, \sigma\, a} J_1(\sigma a)\, J_1(\rho a) - \frac{1}{\sigma} J_1(\sigma a)\, J_o(\rho a) \right\} G \right] \tag{21d}$$

These equations are sufficient for the power loss computation from a tapered rod.

In evaluating the integral, the end points of the region of integration where $\beta = \pm k$ are singular because $\rho^2 = (k^2 - \beta^2)$ will be zero at both end points (see formula 18). But this singularity is only apparent and not real for

$$\lim_{\beta \to \pm k} \frac{|q|^2}{\rho} = \lim_{\beta \to \pm k} \frac{|p|^2}{\rho} = 0$$

The calculation is tedious, but as $\beta \to k$, $F/G \to \sqrt{\frac{\mu_o}{\varepsilon_o}}$ (from 8) and, for the even modes ($F/G > 0$), from 11, we can show

$$\lim_{\beta \to k} \frac{F \ln \rho}{\sqrt{\rho}} = F_o$$

exists, while for the odd modes, $\lim_{\beta \to k} F \cdot \rho^{-5/2}$ exists. For the even modes:

$$L_1 = \lim \rho^{-5/2} \Big\{ (\beta_o + \beta) \gamma \rho \omega \varepsilon_o \frac{\partial H}{\partial a} \left[a \frac{\gamma J_o(\rho a) + \rho H^* J_1(\rho a)}{\gamma^2 + \rho^2} - \frac{J_1(\rho a)}{\rho \gamma} \right]$$

$$+ (k^2 + \beta_o \beta) \frac{\partial K}{\partial a} J_1 (\rho a) \Big\} A$$

$$= \dot{F}_o \, a \, \sigma_o \, a \, J_o (\sigma_o a) (\beta_o + k) k \sqrt{\varepsilon_o / \mu_o} A$$

$$\sigma_o = \sqrt{n^2 - 1}\, k \quad ; \quad F_o = \lim_{\beta \to k} \frac{F \ln \rho}{\sqrt{\rho}}$$

$$L_2 = \lim_{\beta \to k} \rho^{-5/2} \Big\{ (\beta_o + \beta) \gamma \rho \omega \varepsilon_o \frac{\partial I}{\partial a} \left[a \frac{\gamma Y_o(\rho a) + \rho H^* Y_1(\rho a)}{\gamma^2 + \rho^2} - \frac{Y_1(\rho a)}{\gamma \rho} \right]$$

$$+ (k^2 + \beta_o \beta) \frac{\partial M}{\partial a} Y_1 (\rho a) \Big\} A = - L_1$$

$$L_3 = \lim_{\beta \to k} \rho^{-5/2} \Big\{ (\beta_o + \beta) \gamma \rho \omega \mu_o \frac{\partial K}{\partial a} \left[a \frac{\gamma J_o (\rho a) + \rho H^* J_1 (\rho a)}{\gamma^2 + \rho^2} - \frac{J_1 (\rho a)}{\rho \gamma} \right]$$

$$+ (k^2 + \beta_o \beta) J_1 (\rho a) \frac{\partial H}{\partial a} \Big\} B$$

$$= F_o\, a\, (\beta_o + k)k\, \sigma_o\, a\, J_o\, (\sigma_o a)\, B$$

$$L_4 = \lim \rho^{-5/2} \left\{ (\beta_o + \beta)\, \gamma\, \rho\, \omega\, \mu_o \frac{\partial M}{\partial a} \left[a \frac{\gamma\, Y_o\, (\rho a) + \rho\, H^*\, Y_1\, (\rho a)}{\gamma^2 + \rho^2} - \frac{Y_1\, (\rho a)}{\gamma\, \rho} \right] \right.$$

$$\left. + (k^2 + \beta_o\, \beta) \frac{\partial I}{\partial a} Y_1\, (\rho a) \right\} B = - L_3$$

$$\lim_{\beta \to k} \frac{I\, (\rho, Z)}{\sqrt{\rho}} = 0$$

For the odd modes, the same result holds, but its derivation is easier. At the other end point of the region ($\beta = -k$), the even modes have:

$$\lim_{\beta \to -k} F\, \rho^{-5/2} = F_2$$

while the odd modes have

$$\lim_{\beta \to -k} \frac{F\, \ln\rho}{\sqrt{\rho}} = F_3$$

Then, for the even modes, it is easy to show that $\lim_{\beta \to k} \frac{I\, (\rho, Z)}{\rho} = 0$ while the odd modes yield the same result after much difficulty, although the calculation is substantially the same as for the even modes near $\beta = k$.

There is another apparent singularity at $\beta = 0$ where formula 11 gives an infinite value of F. But this, too, is not a real singularity; for, as $\beta \to 0$:

$$\sigma \to nk \quad , \rho \to k$$

and all three limits: $\lim_{\beta \to 0} F/G$, $\lim_{\beta \to 0} \sqrt{|\beta|}\, F$ and $\lim_{\beta \to 0} \sqrt{|\beta|}\, G$ exist. Hence the limits:

$\lim \sqrt{|\beta|}\, \frac{\partial H}{\partial a}$, $\lim \sqrt{|\beta|}\, \frac{\partial I}{\partial a}$, $\lim \sqrt{|\beta|}\, \frac{\partial K}{\partial a}$ and $\lim \sqrt{|\beta|}\, \frac{\partial M}{\partial a}$ all exist for both choices of the sign of F/G and so too does $\lim \sqrt{|\beta|}\, I(\rho, z)$. So the integrand $|\beta|\, I^2/\rho, z)$ has a finite limit at $\beta = 0$ for both even and odd modes.

PART B: NUMERICAL PROCEDURES

A computer program was written to calculate, from Equation A-17, the losses of the dominant mode at a step. The first requirement is to solve the eigenvalue Equation (A-5) for γ. The eigenvalues were found to be very small in the range of interest so the functions of Equation A-5 were expanded in a power series in x and lnx ($x = \gamma a$) and terms involving x^n for $n > 2$ were ignored (see Appendix A). This led to the approximate formula:

$$\gamma a \sim 1.123 \exp \left[-.5 \frac{n^2 + 1}{\kappa_o a} \frac{J_o (\kappa_o a)}{J_1 (\kappa_o a)} \right] \tag{1}$$

Where $\kappa_o = \sqrt{n^2 - 1}\, k$. This formula seems to be valid as long as $\kappa_o a < 2.4048$, the first zero of $J_o(x)$. This value (γ_e) was used as an initial guess and a step of 10% of γ_e was used. The difference in the two sides of the eigenvalue equation was formed for several values of γ starting with $.7 * \gamma_e$ and increasing in steps of $.1 * \gamma_e$ until a sign change was produced. A Regula Falsi method was then used to determine precisely the value of γ. The values determined for $f = 2.75 \times 10^{10}$ and $n^2 = 2.05$ are plotted against k*a for the range $.7 \leq k*a \leq 3.0$ in Figure 1. The final values of γ are compared

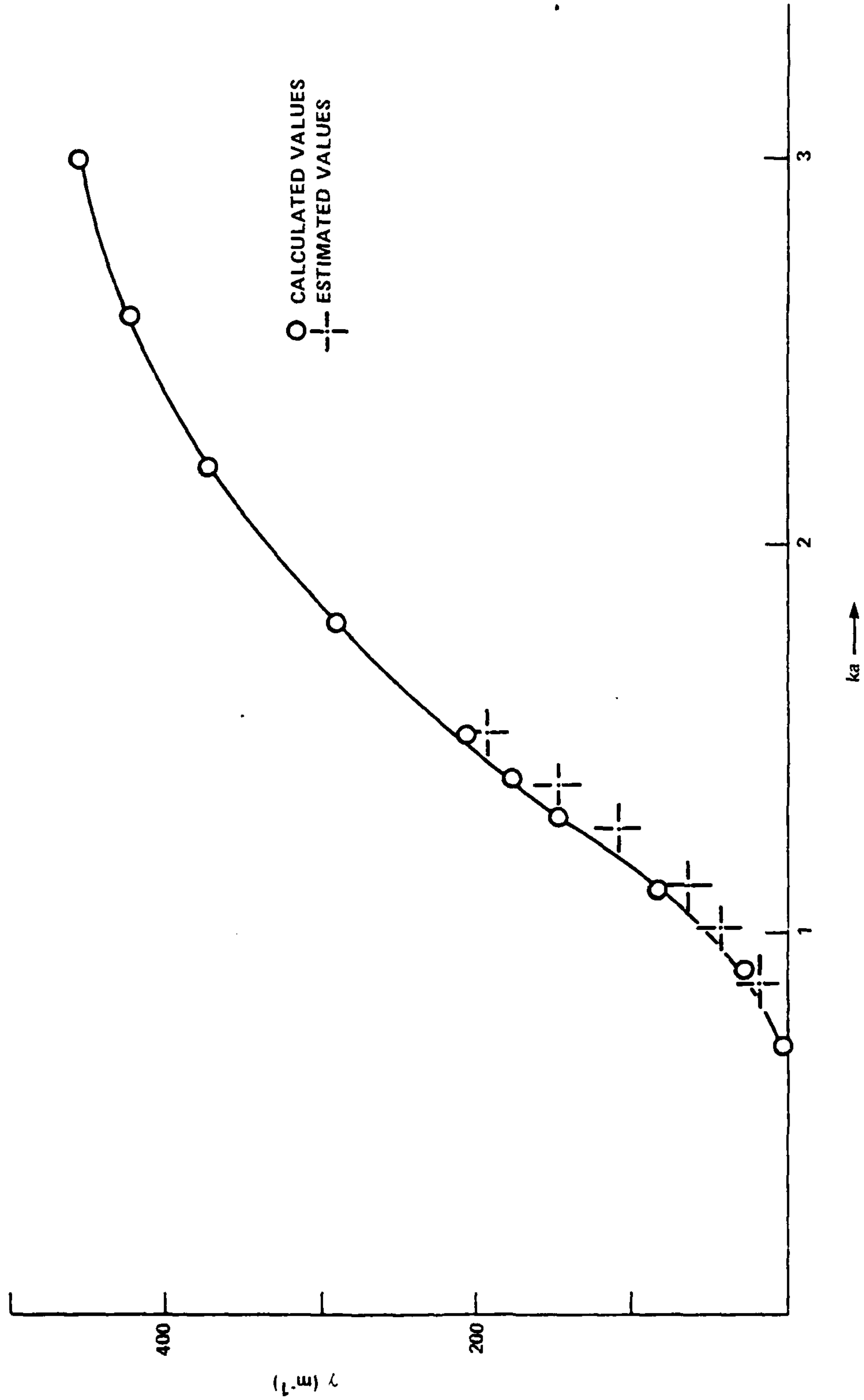

FIGURE 1 γ VS ka

with the values estimated from the simple formula (1) in Appendix A. The agreement is strikingly good (all things considered) for values of $k^*a \leq 1$.

With the value of γ available, β_o and κ can be calculated:

$$B/A = -\sqrt{\frac{\varepsilon_o}{\mu_o}}\,\frac{ka\,(\kappa a)^2}{\beta_o a\left(1+\kappa^2/\gamma^2\right)}\left\{\frac{n^2}{\kappa a}\left[\frac{J_o(\kappa a)}{J_1(\kappa a)}-\frac{1}{\kappa a}\right]\right.$$

$$\left.+\frac{1}{\gamma a}\left[\frac{i\,H^1_o(i\gamma a)}{H^1_1(i\gamma a)}-\frac{1}{\gamma a}\right]\right\} \tag{2}$$

This ratio was calculated and turns out to be very close to $\sqrt{\varepsilon_o/\mu_o}$ for $.7 \leq k^*a \leq 3$. In the expanded scale of Figure 2, the differences are noticeable but they are less than 1%. With the aid of B/A, it is now possible to compute A from the formula:

$$P = \frac{\pi}{4}\left[\frac{k\,\beta_o}{\kappa^4}\left\{(a\kappa)^2\left[J^2_o(\kappa a)+J^2_1(\kappa a)\right]-2\,J^2_1(\kappa a)\right\}\left(n^2+\frac{\mu_o}{\varepsilon_o}\frac{B^2}{A^2}\right)\right.$$

$$+\frac{k\beta_o}{\gamma^4}\left\{(a\gamma)^2\left[\left(\frac{H^1_o(i\gamma a)}{H^1_1(i\gamma a)}\right)^2+1\right]+2\right\}J^2_1(\kappa a)\left(1+\frac{\mu_o}{\varepsilon_o}\frac{B^2}{A^2}\right)$$

$$\left.+2\sqrt{\frac{\mu_o}{\varepsilon_o}}\,\frac{B}{A}\left(\frac{\beta^2_o+n^2k^2}{\kappa^4}-\frac{\beta^2_o+k^2}{\gamma^4}\right)J^2_1(\kappa a)\right]\sqrt{\frac{\varepsilon_o}{\mu_o}}\,A^2 \tag{3}$$

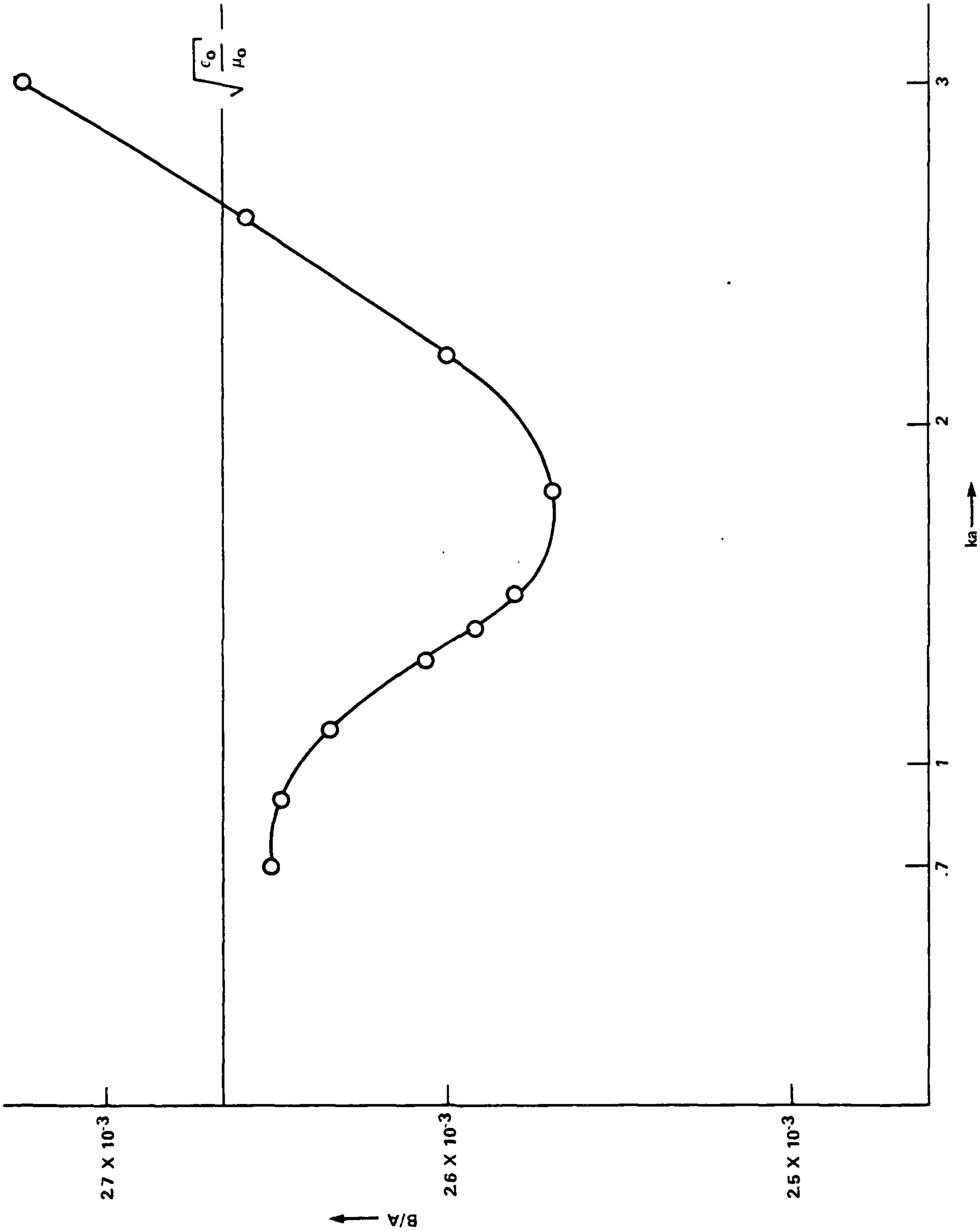

FIGURE 2 RATIO OF MAGNETIC TO ELECTRIC FIELD AMPLITUDE (B/A) VS ka

Assuming unit power in the mode, all the quantities are known except A and the values of A are computed and plotted in Figure 3. A represents the amplitude of the z component of the electric field inside the rod. For very small radii, the amplitude is very small because most of the energy is outside the rod and the inside fields are weak. As the radius increases, the amplitude of E_z increases passing through a peak near $k^*a = 1.8$, then falling off. Now if $k^*a = 1.8$, then D/λ (ratio of rod diameter to radiation wavelength) will be about $1.8/\pi$ or .57. This is just a tad larger than the value of .5 which Yip[2] claims to be the optimum condition for transfer of power to the dielectric rod. Yip's work was for a dielectric constant of 2.56 and, when the calculation is repeated for that value, D/λ turns out to be .51 (approximately).

With the aid of the quantities computed thus far, it is possible to evaluate numerically the two integrals Marcuse gives and from them to compute the transmission and reflection coefficients and, thereby, the power loss:

$$\frac{\Delta P}{P} = 1 - |c_t|^2 - |a_r|^2$$

The values of these for various values of ka are listed in Table 1.

TABLE 1 POWER LOSS AT A STEP

ka_1	c_t	a_r	$\Delta P/P$
1.4	.169	-.0207	.971
1.8	.518	-.0476	.729
2.2	.779	-.0649	.389
2.6	.917	-.0701	.155
3.0	.975	-.0680	.044

Power Loss at a Step From A 13, 14 and 15; $f = 2.75 \times 10^{10}$, $n^2 = 2.05$ and Step Size = $a_2/a_1 = .5$.

[2] Yip, Gar Lam, "Launching Efficiency of the HE_{11} - Surface Wave Mode on a Dielectric Rod." IEEE Transactions on Microwave Theory and Techniques, Vol. MTT-18 #12, Dec 1970.

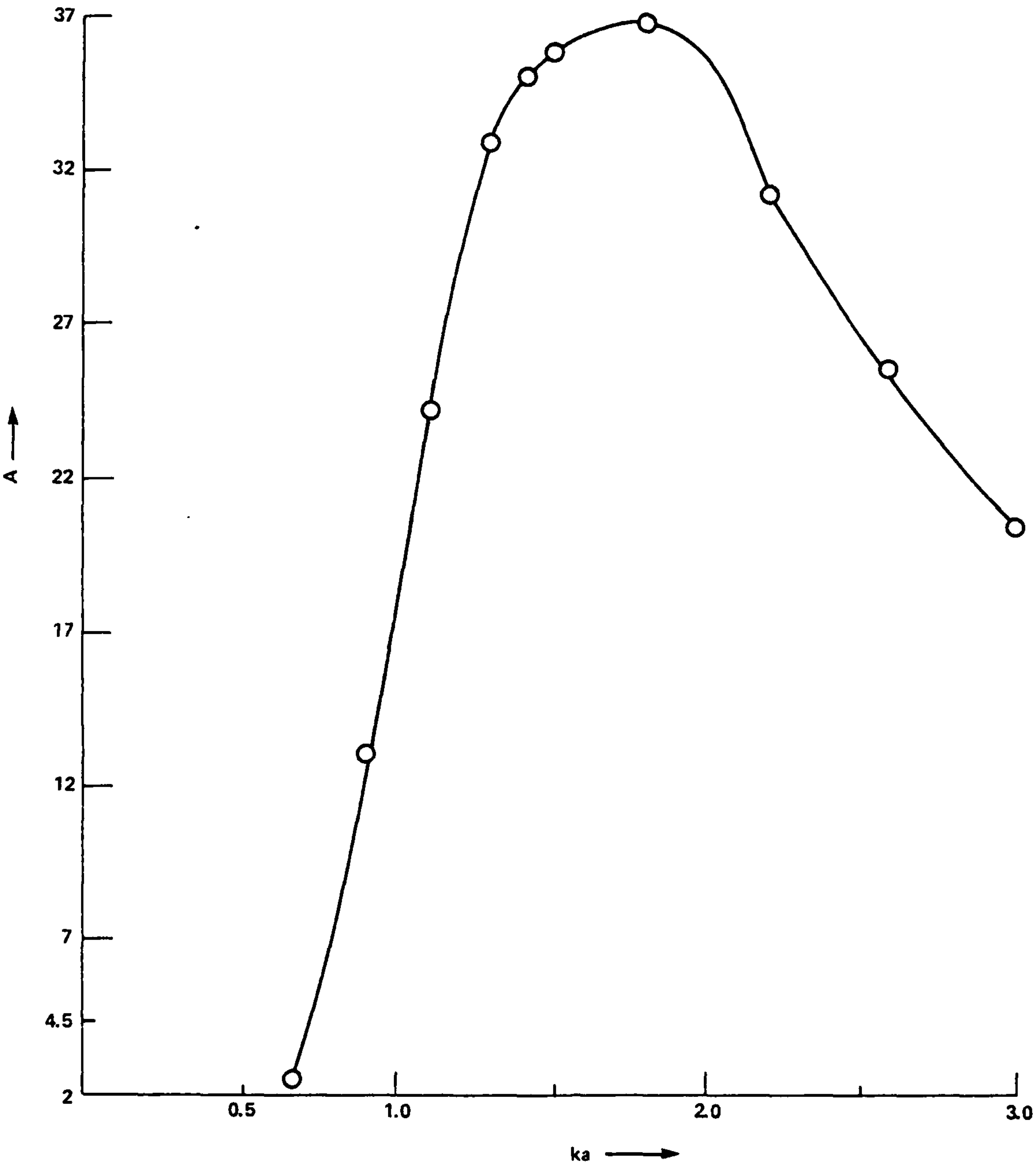

FIGURE 3 AMPLITUDE OF THE Z COMPONENT OF THE ELECTRIC FIELD (UNIT POWER IN THE MODE) VS ka INSIDE THE ROD

From the table, it is clear that the power losses fall off rapidly with increasing values of k*a. Indeed, if k*a $\geq$ 1.8, the power loss varies inversely with the square of k*a (see Figure 4). This will serve as an approximate guide for calculating losses in this range. A complete listing of the program is given in Appendix B. For the tapered dielectric rod, the power loss may be computed from A18-A21. In order to use these formulas, the values of Υ, β_o, A and B must be calculated for each value of the radius, a. In the tapered dielectric, a is a function of z and the first requirement is that a suitable function be chosen. Four different types of function were used and the graphs were (a) linear, (b) exponential, (c) parabolic and (d) elliptic. When the functions were chosen, the axis was divided by 50 points and the radius at each of the fifty points calculated. The eigenvalue problem was solved for each of the points and the values of Υ, A and B were stored in arrays. The ratio F/G was evaluated and both values of F (one for each choice of sign for F/G) were calculated and the corresponding values of G. The partial derivatives $\frac{\partial H}{\partial a}$, $\frac{\partial I}{\partial a}$, $\frac{\partial K}{\partial a}$ and $\frac{\partial M}{\partial a}$ were then evaluated and, finally, I(ρ, z). The β-integral: $\left[\exp -i \int_o^z (\beta_o - \beta)\, ds\right]$ was evaluated using Simpson's rule when z/Δz was even. When z/Δz was odd, Simpson's rule was used to calculate the integral up to z-Δz and the trapezoidal rule to extend the integral to z. With I(ρ, z) and the β-integral both defined, Simpson's rule is used to calculate q(ρ) & p(ρ) from A19 and used again to calculate the power loss from A18.

For the linear tapers, the step was always .5 and the length was varied to make the taper more or less steep. The dielectric constant was also varied to study its effects on the losses. The results are summarized in Table 2.

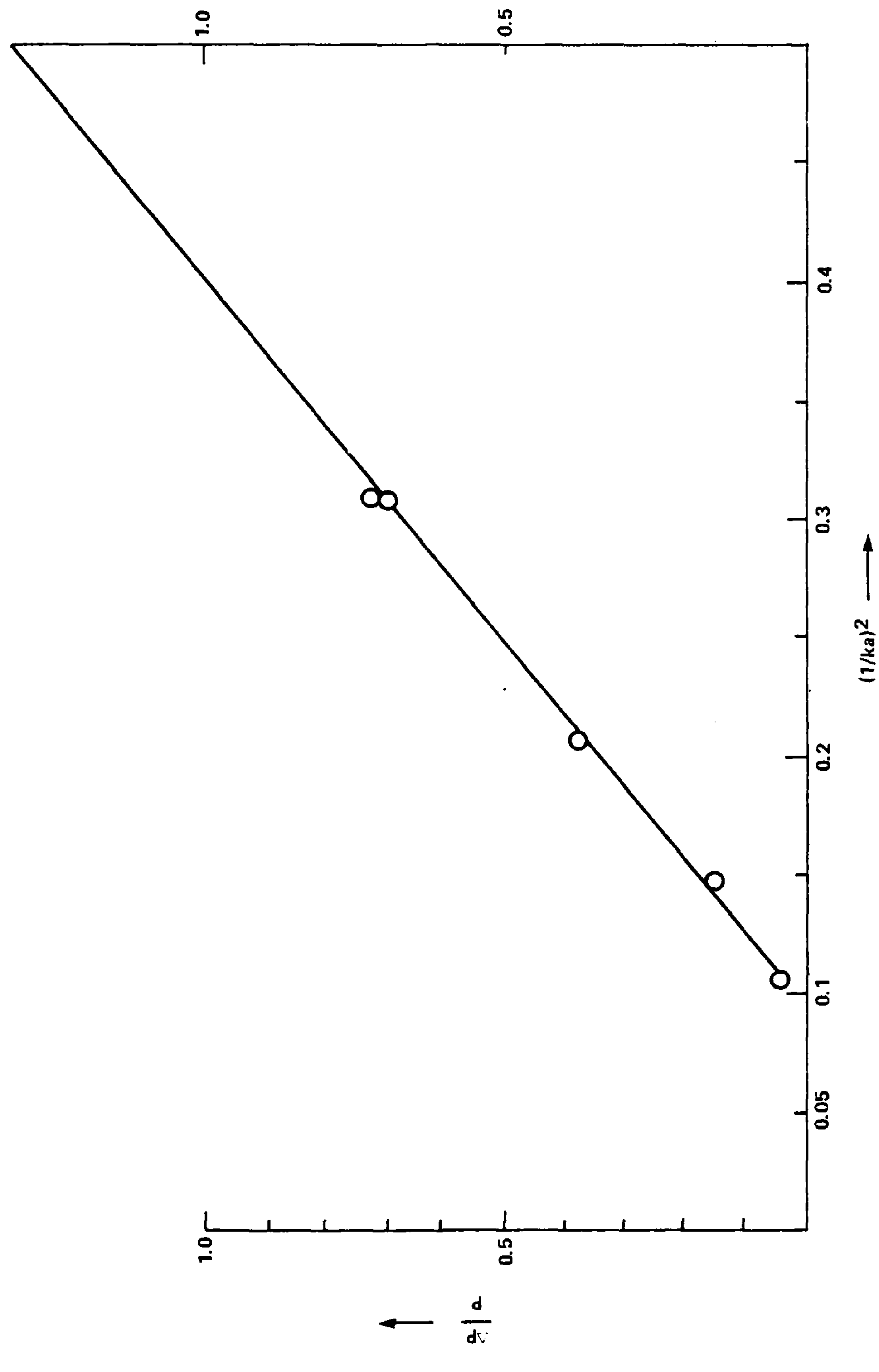

FIGURE 4 POWER LOSS PROPORTIONAL TO $(1/ka)^2$ $1.8 \leq ka \leq 3.0$

TABLE 2 POWER LOSS OF A TAPERED ROD (LINEAR TAPER)

		← n^2 →		
		1.5	2.05	2.56
↑ L/a_1 ↓	10	.42482	.11898	.034436
	20	.30241	.067682	.016090
	100	.045451	.010152	.002784

Power Loss ($\Delta P/P$) as a Function of Dielectric Constant (n^2) and Length to Radius Ratio (L/a_1) for $f = 3 \times 10^{14}$, $a_1 = .4 \times 10^{-6}$, $a_2 = .2 \times 10^{-6}$, $ka_1 = 2.513$.

As is readily seen, long rods lose less power than short rods and high dielectric materials lose less than low ones. For exponential tapers with exponent 4.6 (ie: exp − 4.6* z/L), the results are summarized in Table 3.

TABLE 3 POWER LOSS OF A TAPERED ROD (EXPONENTIAL TAPER)

		n^2		
		1.5	2.05	2.56
L/a_1	10	.45190	.12191	.020800
	20	.36540	.045415	.0025997
	100	.054494	.0012596	.0022469
	50		.0016464	

Power Loss as a Function of Dielectric Constant and Length to Radius Ratio for an Exponentially Tapered Rod. (Frequency & step size as in Table 2)

Comparison of Tables 2 and 3 show that the exponential taper is not very much better than the linear taper in the region considered. The question naturally arises as to whether the exponent −4.6 is the best choice. So a family of exponential curves was chosen (exp − p * z/L) and the dielectric constant was taken to be 2.05. The calculations were then performed for various values of p. If $p \to \infty$, the curve approaches a step and the results of the

preceeding section are applicable. If $p \rightarrow 0$, the curve approaches a straight line and the linear taper calculations apply. The results are summarized in Table 4.

TABLE 4 POWER LOSS OF A TAPERED ROD (VARIOUS EXPONENTS)

		$\leftarrow L/a_1 \rightarrow$				
		3	10	30	100	taper
	0		.119		.010152	Linear
	3	.227	.110	.02437	.00131	$e^{-3z/L}$
$\uparrow$ p $\downarrow$	4.6		.122		.00126	$e^{-4.6z/L}$
	7	.244	.163	.02364	.00230	$e^{-7z/L}$
	∞	.194	.194	.194	.194	STEP

Power Loss for Various Exponential Curves and Various Lengths.
$n^2 = 2.05$.

From the table, it appears that minimum power losses occur for different exponents depending on the value of L/a_1. For $L/a_1 = 10$, the minimizing value of p should be close to 3, while for $L/a_1 = 100$, it should be nearer to 4.6. The dependence of p on the dielectric constant would require further calculation.

Two more tapers were calculated for various values of L/A. The first was a parabolic taper with the vertex at the thin end of the rod and the second was an ellipse whose center was above the thin end of the rod and to the left of the thick end (see Table 5).

TABLE 5 POWER LOSS OF VARIOUS TAPERS

	← L/a_1 →			
Taper	3	10	30	100
Ellipse	.21935	.10793	.021004	.0015239
Parabola	.229	.11975	.036586	.001378

The elliptical taper has lower losses than the parabolic for short antennas and the same is true when the elliptical rod is compared with the exponential and linear tapers.

A complete listing of this program is given in Appendix B.

APPENDIX A

SOLUTION OF THE EIGENVALUE PROBLEM

For the bound modes, it is necessary that β_o satisfy the eigenvalue equation.

$$\left[1 - \gamma a \frac{i\, H_o^1\,(i\gamma a)}{H_1^1\,(i\gamma a)} - n^2 \frac{a\gamma^2}{\kappa}\left(\frac{J_o\,(\kappa a)}{J_1\,(\kappa a)} - \frac{1}{\kappa a}\right)\right] *$$

$$\left[1 - \gamma a \frac{i\, H_o^1\,(i\gamma a)}{H_1^1\,(i\gamma a)} - \frac{a\gamma^2}{\kappa}\left(\frac{J_o\,(\kappa a)}{J_1\,(\kappa a)} - \frac{1}{\kappa a}\right)\right]$$

$$= \left[\frac{(n^2 - 1)\,\beta_o\,k}{\kappa^2}\right]^2$$

where $\gamma = \sqrt{\beta_o^2 - k^2}$ and $\kappa = \sqrt{n^2 k^2 - \beta_o^2} = \sqrt{(n^2 - 1)\,k^2 - \gamma^2}$

Set $\gamma a = x$, $\kappa a = \kappa^*$ and $ka = k^*$

Expand the term $\dfrac{J_o\,(\kappa^*)}{J_1\,(\kappa^*)} - \dfrac{1}{\kappa^*}$ in a power series about $x = 0$ and set t_1

$$t_1 = \frac{J_o\,(\kappa_o^*)}{J_1\,(\kappa_o^*)} - \frac{1}{\kappa_o^*} \quad , \quad \text{where } \kappa_o^* = \sqrt{n^2 - 1}\;k^*$$

Expand $\gamma a \dfrac{i\,H_o^1(i\gamma a)}{H_1^1(i\gamma a)}$ in a power series in powers of x and lnx. The result is:

$$\gamma a \frac{i\,H_o^1(i\gamma a)}{H_1^1(i\gamma a)} = -(\gamma a)^2\left[\gamma_1 + \ln\frac{\gamma a}{2}\right] + O(\gamma a)^2$$

where γ_1 = the Euler constant ≃ .5772

The eigenvalue equation for γa (=x) becomes:

$$\left[1 - x^2\left(\gamma_1 + \ln\frac{x}{2}\right) + \frac{n^2x^2}{\kappa_o^*}t_1\right] *$$

$$\left[1 - x^2\left(\gamma_1 + \ln\frac{x}{2}\right) + \frac{x^2}{\kappa_o^*}t_1\right] =$$

$$\frac{(n^2-1)^2 k^{*2}(k^{*2}+x^2)}{\left[(n^2-1)k^{*2} - x^2\right]^2}$$

$$1 - x^2\left(2\gamma_1 + 2\ln\frac{x}{2} + (n^2+1)\frac{t_1}{\kappa_o^*}\right) + O(x^4) = \frac{1 + \dfrac{x^2}{k^{*2}}}{\left[1 - \dfrac{x^2}{(n^2-1)k^{*2}}\right]^2}$$

$$1 - x^2\left(2\gamma_1 + 2\ln\frac{x}{2} + (n^2+1)\frac{t_1}{\kappa_o^*}\right) + O(x^4) = \left(1 + \frac{x^2}{k^{*2}}\right)\left(1 - \frac{x^2}{(n^2-1)k^{*2}}\right)^{-2}$$

$$= \left(1 + \frac{x^2}{k^{*2}}\right)\left(1 + \frac{2x^2}{(n^2-1)k^{*2}}\right) + O(x^4)$$

$$-x^2\left(2\gamma_1 + 2\ln\frac{x}{2} + (n^2+1)\frac{t_1}{\kappa_o^*}\right) = \frac{x^2}{k^{*2}} + \frac{2x^2}{(n^2-1)k^{*2}} + O(x^4)$$

$$-\left(2\gamma_1 + 2\ln\frac{x}{2} + (n^2+1)\frac{t_1}{\kappa_o^*}\right) = \frac{1}{k^{*2}}\left[1+\frac{2}{(n^2-1)}\right] + O(x^2)$$

$$2\gamma_1 + 2\ln\frac{x}{2} + \frac{(n^2+1)t_1}{\kappa_o^*} = -\frac{1}{k^2}\left[\frac{n^2+1}{n^2-1}\right]$$

$$2\ln\frac{x}{2} = -\frac{1}{k^2}\left(\frac{n^2+1}{n^2-1}\right) - 2\gamma_1 - \frac{n^2+1}{\kappa_o^*}t_1$$

$$= -\left[\frac{1}{k^{*2}}\frac{n^2+1}{n^2-1} + 2\gamma_1 + \frac{n^2+1}{\kappa_o^*}\frac{J_o(\kappa_o^*)}{J_1(\kappa_o^*)} - \frac{n^2+1}{\kappa_o^{*2}}\right]$$

$$\kappa_o^{*2} = (n^2-1)k^{*2}$$

$$\frac{n^2+1}{\kappa_o^{*2}} = \frac{n^2+1}{n^2-1}\frac{1}{k^{*2}} \Rightarrow \ln\frac{x}{2} = -\frac{1}{2}\left\{2\gamma_1 + \frac{n^2+1}{\kappa_o^*}\frac{J_o(\kappa_o^*)}{J_1(\kappa_o^*)}\right\}$$

$$\ln\frac{x}{2} \cong -\left[\gamma_1 + \frac{n^2+1}{2}\frac{1}{\kappa_o^*}\frac{J_o(\kappa_o^*)}{J_1(\kappa_o^*)}\right] + O(x^2)$$

$$x \approx 2\exp(-\gamma_1)\exp\left\{-\frac{n^2+1}{2}\frac{1}{\kappa_o^*}\frac{J_o(\kappa_o^*)}{J_1(\kappa_o^*)}\right\}$$

$$\gamma \sim \frac{2}{a} \exp(-.5772) \exp - \frac{n^2+1}{2} \frac{1}{\sqrt{n^2-1}} \frac{1}{ka} \frac{J_o\left(\sqrt{n^2-1}\, ka\right)}{J_1\left(\sqrt{n^2-1}\, ka\right)}$$

$$\gamma a \sim 1.123 \exp - \frac{1}{2} \frac{n^2+1}{\kappa_o a} \frac{J_o(\kappa_o a)}{J_1(\kappa_o a)}$$

This formula gives an estimate for the value of γ that satisfies the eigenvalue equation. The estimate will be reliable when (1) $\gamma a \ll 1$ and (2) $\sqrt{n^2 - 1}\, ka <$ the first zero of $J_o \simeq 2.4$

TABLE A-1 SOLUTION OF THE EIGENVALUE PROBLEM

For $n^2 = 2.05$		$ka < \frac{2.4}{1.05} = 2.34$
ka	γa (formula)	γa (computer calculation)
.5	1.5×10^{-5}	1.48×10^{-5}
.625	9.74×10^{-4}	9.76×10^{-5}
.75	9.45×10^{-3}	9.50×10^{-3}
.875	3.77×10^{-2}	3.77×10^{-2}
1.0	1.08×10^{-1}	9.32×10^{-2}
1.125	1.69×10^{-1}	
1.25	2.63×10^{-1}	
1.375	3.56×10^{-1}	
1.5	4.72×10^{-1}	5.48×10^{-1}

For the range of values in Table A-1, the formula seems to be accurate to within 15%. It gives a useful initial guess for the start of an iteration scheme.

APPENDIX B

FORTRAN LISTING OF THE COMPUTER PROGRAM USED

```
      PROGRAM TAPER(INPUT,OUTPUT,TAPE6)
      COMMON  A,       BETA,    BETA0,   BINT,    CAPA,    CAPB,
     2DHOA(2),DIDA(2),DKDA(2), DMDA(2), EPS0 ,   F(2),    FOG(2),
     3FUNC,   G(2),    GAMMA,   IRZ1,    IRZ2,    J0R,     J0S,
     4 J1R,   J1RP,    J1S,     J1SP,    K,       KAPPA,   KSQ,
     5LENGTH,  MU,      MU0,     NSQ,     OMEGA,   P,       PI,
     6 RHO,   SIGMA,   Y0R,     Y0S,     Y1R,     Y1RP,    Y1S,
     7 Y1SP
      COMMON /NUMBER/ KMAX,NMAX,IPT,IPTM1,AE,ALEN,SWITCH,Z,DZ
      COMMON /ARY/ AARRAY
      COMMON /BRY/ BARRAY
      COMMON /GRY/ GARRAY
      COMMON/BIN/BINT1
      DIMENSION BINT1(51)
      DIMENSION          GARRAY( 51 ),AARRAY( 51 ),BARRAY( 51 )
      REAL LENGTH,K,KSQ,NSQ,LOA   ,MU,MU0,NU,KAPPA
C     ALL DIMENSIONS MUST BE MKS     $$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$$
      P=1.
      PI=3.14159
      MU0=4.E-7*PI $ EPS0=1.E-9/36./PI
      IPT =144     $ IPTM1 =IPT-1
       NMAX =  50  $ KMAX =50
      NMAX1 = NMAX +1
    3 READ 5,A,LOA,STEP,NU,NSQ          ,SWITCH
    5 FORMAT(6E12.5)
      IF(LOA.LE.0.) STOP
      OMEGA =2.*PI*NU
      AE = STEP*A
      LENGTH = LOA *A
      PRINT 6,A,AE,LOA,NU,NSQ
    6 FORMAT(*1THE GEOMETRY IS*,/,* BEGINNING RADIUS = *,E12.5,
     C *FINAL RADIUS = *,E12.5,/,* LENGTH = *,E12.5,//,* FREQUENCY = *,
     A E12.5,* DIELECTRIC CONSTANT = *,E12.5)
      ALEN = LENGTH
      MU= INIT(A)
      MU=MU0
      KSQ =OMEGA*OMEGA*MU0*EPS0 $ K=SQRT(KSQ)
      CALL GAMSOLV
      CALL BINTGRL
      DINT=0.
      SIMPSON =4. $ SIGN =-1.
      BETA =-K $ DBETA =K/FLOAT(KMAX )
      KMAX = 2* KMAX  -1
      DO 100 KK=1, KMAX
      BETA = BETA + DBETA $ RHO =SQRT(KSQ-BETA*BETA)
      IF(ABS(BETA).GT.0.05*DBETA) GO TO         98
      CALL BETEQ0
      DINT = DINT+ FUNC
      GO TO 99
   98 SIGMA = SQRT(NSQ*KSQ-BETA*BETA)
      CALL INTEG
      DINT = DINT+ SIMPSON*FUNC*ABS(BETA)/RHO
   99 SIMPSON = SIMPSON +2.*SIGN
  100 SIGN = -SIGN
      DPOP =  DINT*DBETA/3.
      PRINT 12,DINT,DBETA,DPOP
   12 FORMAT(1H1, E12.5,*DBETA=*,E15.9,*AND THE POWER LOSS IS*,E12.5,//)
      KMAX = (KMAX +1)/2  $ GOTO 3
      STOP $ END
      SUBROUTINE BINTGRL
      COMMON  A,       BETA,    BETA0,   BINT,    CAPA,    CAPB,
     2DHOA(2),DIDA(2),DKDA(2), DMDA(2), EPS0 ,   F(2),    FOG(2),
     3FUNC,   G(2),    GAMMA,   IRZ1,    IRZ2,    J0R,     J0S,
     4 J1R,   J1RP,    J1S,     J1SP,    K,       KAPPA,   KSQ,
```

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```
     5LENGTH, MU,       MUO,      NSQ,      OMEGA,    P,        PI,
     6 RHO,    SIGMA,   YOR,      YOS,      Y1R,      Y1RP,     Y1S,
     7 Y1SP
      COMMON /NUMBER/ KMAX,NMAX,IPT,IPTM1,AE,ALEN,SWITCH,Z,DZ
      COMMON /BIN/ BINT1
      COMMON/GRY/GARRAY
      DIMENSION GARRAY(1 ),BINT1(1 )
      REAL IRZ1,IRZ2,JOR,JOS,J1R,J1S,J1RP,J1SP,K,KAPPA,KSQ,MU,MUO,NSQ
      REAL LENGTH
C     USES A COMBINATION OF SIMPSCN*S RULE AND THE TRAPEZOIDAL RULE TC EVALU
C     THE INTEGRALS
      BINT1(1)=0.
      FIRST = SQRT(KSQ+GARRAY(1)*GARRAY(1)    )
      DC 5 NN=2,NMAX ,2
      SECOND = SQRT(KSQ+ GARRAY(NN)*GARRAY(NN) )
      BINT1(NN) = BINT1(NN-1) +.5*(FIRST+SECOND)*DZ
      NNP = NN+1
      SECOND = FIRST +4.*SECOND
      FIRST = SQRT(KSQ+ GARRAY(NNP)*GARRAY(NNP) )
    5 BINT1(NNP)= BINT1(NN-1) +(FIRST+SECOND)*DZ/3.
      RETURN $ END
      SUBROUTINE INTEG
      COMMON  A,        BETA,     BETAO,    BINT,     CAPA,     CAPB,
     2DMDA(2),DIDA(2),DKDA(2),  DMDA(2),  EPSO ,    F(2),     FOG(2),
     3FUNC,    G(2),     GAMMA,    IRZ1,     IRZ2,     JOR,      JOS,
     4 J1R,    J1RP,     J1S,      J1SP,     K,        KAPPA,    KSQ,
     5LENGTH, MU,       MUO,      NSQ,      OMEGA,    P,        PI,
     6 RHO,    SIGMA,   YOR,      YOS,      Y1R,      Y1RP,     Y1S,
     7 Y1SP
      COMMON /NUMBER/ KMAX,NMAX,IPT,IPTM1,AE,ALEN,SWITCH,Z,DZ
      COMMON /BIN/ BINT1
      DIMENSION BINT1(1)
      REAL JOR,JOS,J1R,J1RP,J1S,J1SP,K,KAPPA,KSQ,LENGTH,MU,MUO,NSQ
      REAL      IRZ1,IRZ2
      COMPLEX ARG,SUM1,SUM2
      PRINT 2, BETA
    2 FORMAT( 1H1,* BETA =*, E12.5,*  $$$$$$$$$$$$$$* )
      IPRT=1
      NMAX 1= NMAX +1
      Z=0
      A=FN(Z) $ DADZ= FNP(Z) $ P=1.
      CALL GAMFIND
      CALL ERROR
      CALL IRZC
      PRINT 3,A,Z,IRZ1,IRZ2
      SUM1 = IRZ1*DADZ*BEJ1(KAPPA*A) /GAMMA/GAMMA
      SUM2 = IRZ2*DADZ*BEJ1(KAPPA*A)/GAMMA/GAMMA
      SIMPSON =4.   $ SIGN =-1.
      DZ = LENGTH/FLOAT(NMAX) $ Z= DZ$NMAXM1=NMAX-1
      DC 100 NN=1,NMAXM1   $A=FN(Z) $ DADZ=FNP(Z)
      CALL GAMFIND
      BINT = BINT1(NN+1) -BETA*Z
      ARG = CMPLX(0.,BINT) $ ARG=CEXP(-ARG)
      CALL IRZC
      CALL ERROR
      IF(IPRT.LE.1)
     1PRINT 3,A,Z,IRZ1,IRZ2,BINT  ,GAMMA  ,CAPA,CAPB
    3 FORMAT(1H ,*A=*,E12.5,* Z=*,F12.5,*    * , 6E12.5)
      IF(IPRT.GT.IPTM1) IPRT = IPRT -IPT
      IPRT = IPRT +1
      SUM1=SUM1+ IRZ1*DADZ*ARG*SIMPSCN *BEJ1(KAPPA*A)/GAMMA/GAMMA
      SUM2=SUM2+ IRZ2*DADZ*ARG*SIMPSCN *BEJ1(KAPPA*A)/GAMMA/GAMMA
      SIMPSON = SIMPSON +2.*SIGN $ SIGN =-SIGN
  100 Z=Z+DZ
```

```
 A=FN(Z) $ DADZ = FNP(Z)
 CALL GAMFIND
 BINT = BINT1(NMAX1) - BETA*Z
 ARG=CMPLX(0.,BINT)    $ ARG =CEXP(-ARG)
 CALL IRZC
 PRINT 3,A,Z,IRZ1,IRZ2
 SUM1= SUM1+IRZ1*DADZ*ARG *BEJ1(KAPPA*A)/GAMMA/GAMMA
 SUM2= SUM2+IRZ2*DADZ*ARG*BEJ1(KAPPA*A) /GAMMA/GAMMA
 FACTOR=.25*PI              /RHC/RHO
 SUM1= SUM1*FACTOR $ SUM2=SUM2*FACTOR
 SUM1=SUM1*DZ/3. $ SUM2=SUM2*DZ/3.
 FUNC= CABS(SUM1)**2 + CABS(SUM2)**2
 PRINT 4,FUNC
4 FORMAT(* THE VALUE OF Q SQUARED  + P SQUARED FOR THIS VALUE OF BET
 TA IS*, E20.12)
 RETURN $ END
 SUBROUTINE IRZC
 COMMON  A,          BETA,     BETA0,    BINT,     CAPA,     CAPB,
2DHDA(2),DIDA(2),DKDA(2),  DMDA(2), EPS0 ,   F(2),     FOG(2),
3FUNC,    G(2),      GAMMA,    IRZ1,     IRZ2,     J0R,      J0S,
4 J1R,    J1RP,      J1S,    - J1SP,     K,        KAPPA,    KSC,
5LENGTH, MU,         MU0,      NSQ,      OMEGA,    P,        PI,
6 RHO,    SIGMA,     Y0R,      Y0S,      Y1R,      Y1RP,     Y1S,
7 Y1SP
 DIMENSION     H(3), HF(3)
 DIMENSION IRZ(2)
 REAL IRZ1,IRZ2
 REAL IRZ ,MU
 REAL J0S,J1S,J1SP,J0R,J1R,J1RP,NSQ,K,MU0,MU,KSQ,KAPPA
 CALL HANKEL(1,GAMMA*A,HF,H,F0,H)
 CALL FOGC
 TERM = A*(GAMMA*J0R-RHO*F0/HF(1)*J1R)/(GAMMA*GAMMA+RHO*RHO)
 TERM= TERM-J1R/RHC /GAMMA
 TERM1= A*(GAMMA*Y0R-RHO*F0*Y1R/HF(1) )
 TERM1=TERM1/(GAMMA*GAMMA+RHO*RHO)-Y1R/GAMMA/RHO
 CALL DHDAC $ CALL DKDAC
 CALL DIDAC $ CALL DMDAC
 FAC1=(BETA0+BETA)*GAMMA*RHO*OMEGA
 FAC2= FAC1*MU*CAPB$FAC1=FAC1*EPS0*CAPA
 FAC1= FAC1*TERM+(KSQ+BETA0*BETA)*CAPB*J1R
 FAC2= FAC2*TERM+(KSQ+BETA0*BETA)*CAPA *J1R
 FAC3=(BETA0+BETA)*GAMMA*RHO*OMEGA
 FAC4=FAC3*MU*CAPB*TERM1  $ FAC3= FAC3*EPS0*CAPA*TERM1
 FAC3= FAC3+Y1R*CAPB*(KSQ+BETA0*BETA)
 FAC4= FAC4+Y1R*CAPA*(KSQ+BETA0*BETA)
 DO 2 II=1,2
2 IRZ(II)=FAC1*DHDA(II)+FAC2*DKDA(II)+FAC3*DIDA(II)+FAC4*DMDA(II)
 IRZ1 = IRZ(1)        $ IRZ2= IRZ(2)
 RETURN $ END
 SUBROUTINE BETEQ0
 COMMON  A,          BETA,     BETA0,    BINT,     CAPA,     CAPB,
2DHDA(2),DIDA(2),DKDA(2),  DMDA(2), EPS0 ,   F(2),     FOG(2),
3FUNC,    G(2),      GAMMA,    IRZ1,     IRZ2,     J0R,      J0S,
4 J1R,    J1RP,      J1S,      J1SP,     K,        KAPPA,    KSC,
5LENGTH,  MU,        MU0,      NSC,      OMEGA,    P,        PI,
6 RHO,    SIGMA,     Y0R,      Y0S,      Y1R,      Y1RP,     Y1S,
7 Y1SP
 COMMON /BIN/ BINT1
 DIMENSION BINT1(1)
 COMMON /NUMBER/ KMAX,NMAX,IPT,IPTM1,AE,ALEN,SWITCH,Z,DZ
 REAL LENGTH,NSQ
 REAL IRZ1,IRZ2,J0R,J0S,J1R,J1S,J1RP,J1SP,K,KAPPA,KSQ,MU,MU0,NSQ
 COMPLEX ARG,SUM1,SUM2
 PRINT 2, BETA
```

```
 2 FORMAT(1H1,* BETA = *,E12.5,*      $$$$$$$$$$$$$$$*)
   IPRT=1
   BETA =0.
   SIGMA =K*SQRT(NSQ)
   Z =0.
   A = FN(Z) $ DADZ= FNP(Z)
   NMAX1 = NMAX +1
   CALL GAMFIND
   CALL IRZ0
   PRINT 3,A,Z,IRZ1,IRZ2
 3 FORMAT(1H ,*A= *,F12.5,*Z= *,E12.5,*    *,6E12.5)
   SUM1 =IRZ1*DADZ*BEJ1(KAPPA*A)/GAMMA/GAMMA
   SUM2 =IRZ2*DADZ*BEJ1(KAPPA*A)/GAMMA/GAMMA
   SIMPSON =4. $ SIGN =-1
   Z = DZ
   NMAXM1 = NMAX-1
    DO 100 NN =1,NMAXM1
   A = FN(Z) $ DADZ = FNP(Z)
   CALL GAMFIND
   BINT = BINT1(NN+1)
   ARG =CMPLX(0. ,BINT) $ ARG = CEXP(-ARG)
   CALL IRZ0
   CALL ERROR
   IF(IPRT.LE.1)
  1PRINT 3,A,Z,IRZ1,IRZ2,BINT,GAMMA,CAPA,CAPB
   IF(IPRT.GT.IPRTM1) IPRT = IPRT -IPT
   IPRT = IPRT +1
   SUM1 = SUM1 + IRZ1*DADZ*ARG*SIMPSON*BEJ1(KAPPA*A)/GAMMA/GAMMA
   SUM2 = SUM2 + IRZ2*DADZ*ARG*SIMPSON*BEJ1(KAPPA*A)/GAMMA/GAMMA
   SIMPSON = SIMPSON +2.*SIGN
    SIGN = -SIGN
100 Z= Z + DZ
   A = FN(Z) $ DADZ = FNP(Z)
   CALL GAMFIND
   BINT = BINT1(NMAX1)
   ARG =CMPLX(0. ,BINT) $ ARG = CEXP(-ARG)
   CALL IRZ0
   PRINT 3,A,Z,IRZ1,IRZ2
   SUM1 = SUM1 + IRZ1*DADZ*ARG          *BEJ1(KAPPA*A)/GAMMA/GAMMA
   SUM2 = SUM2 + IRZ2*DADZ*ARG          *BEJ1(KAPPA*A)/GAMMA/GAMMA
   FACTOR = .25*PI/RHO/RHO*DZ/3.
   SUM1 = SUM1*FACTOR  $ SUM2= SUM2*FACTOR
   FUNC = CABS(SUM1)**2 + CABS(SUM2)**2
   FUNC = FUNC/RHO
   RETURN $ END
   SUBROUTINE BESSFN
   COMMON /GAMSOL/NMAX
   COMMON  A,         BETA,     BETA0,    BINT,     CAPA,     CAPB,
  2DHDA(2),DIDA(2),DKDA(2),   DMDA(2),  EPS0 ,    F(2),     FOG(2),
  3FUNC,    G(2),     GAMMA,    IRZ1,     IRZ2,     J0R,      J0S,
  4 J1R,    J1RP,     J1S,      J1SP,     K,        KAPPA,    KSQ,
  5LENGTH,   MU,       MU0,      NSQ,      OMEGA,   P,         PI,
  6 RHO,    SIGMA,    Y0R,      Y0S,      Y1R,      Y1RP,     Y1S,
  7 Y1SP
   REAL IRZ1,IRZ2,J0R,J0S,J1R,J1RP,J1S,J1SP,K,KAPPA,KSQ,LENGTH,MU,MU0
   REAL NSQ
   R = RHO*A   $ S= SIGMA*A
   J0R=BEJ0(R)   $ Y0R=BEY0(R)
   J1R=BEJ1(R)   $ Y1R=BEY1(R)
   J1RP = (R*J0R-J1R)/R    $ Y1RP =(R*Y0R-Y1R)/R
   J0S=BEJ0(S)   $ Y0S=BEY0(S)
   J1S=BEJ1(S)   $ Y1S=BEY1(S)
   J1SP = (S*J0S-J1S)/S    $ Y1SP =(S*Y0S-Y1S)/S
   RETURN $ END
```

```
 SUBROUTINE IRZ0
   COMMON  A,       BETA,     BETA0,    BINT,     CAPA,     CAPB,
2DHDA(2),DIDA(2),DKDA(2),  DMDA(2), EPS0 ,    F(2),     FOG(2),
3FUNC,    G(2),     GAMMA,    IRZ1,     IRZ2,     J0R,      J0S,
4 J1R,    J1RP,     J1S,      J1SP,     K,        KAPPA,    KSQ,
5LENGTH,   MU,       MU0,      NSQ,      OMEGA,    P,        PI,
6 RHO,    SIGMA,    Y0R,      Y0S,      Y1R,      Y1RP,     Y1S,
7 Y1SP
 COMMON /NUMBER/ KMAX,NMAX,IPT,IPTM1,AE,ALEN,SWITCH,Z,DZ
 DIMENSION HF(3),DUM(3) , IRZ(2)
 REAL J0R,J0S,J1R,J1S,J1RP,J1SP,IRZ,K,KSQ,MU0,NSQ
 REAL IRZ1,IRZ2,KAPPA,LENGTH,MU
 RHO =K
 SIGMA =K*SQRT(NSQ)
 CALL BESSFN
 B = K*Y1R*J1SP/SIGMA
 D=K*J1R*J1SP/SIGMA
 E=J1S*J1RP
 GG=J1S*Y1RP
 F1= SQRT(MU0/EPS0)
 FOG(1) =(GG-B)*(GG-B) +(E-D)*(E-D)
 FOG(1) =FOG(1)/((GG-NSQ*B)**2 +(E-NSQ*D)**2)
 FOG(1) =SQRT(FOG(1) ) *F1
 FOG(2) = - FOG(1)
 FAC = .125*PI**3*A*A*OMEGA*EPS0/RHO
 T1 = (GG-NSQ*E)**2 + (E-NSQ*D)**2
 T2=(GG-B)**2 + (E-D)**2
 T2 = T2*MU0/EPS0/FOG(1)/FOG(1)
 F(1) = FAC*(T1 +T2)
 F(1) =SQRT(1./F(1)    )
 F(2) = F(1)
 G(1) = F(1)/FOG(1)
 G(2) = - G(1)
 DHDA(1) = PI*K*.5*(NSQ-1.)*K*A*F(1)*J1S
 DIDA(1) = -DHDA(1)*J1R
 DHDA(1) = DHDA(1)*Y1R
 DHDA(2) = DHDA(1)
 DIDA(2) = DIDA(1)
 DKDA(1) =.5*PI*(NSQ-1.)*KSQ/SQRT(NSQ)
 T1 = A* J0S
 T2 = 2.*J1S/A/KSQ/SQRT(NSQ)
 T3 = -J1S/K/SQRT(NSQ)
 DMDA(1) = -DKDA(1)*(T1*J1RP+T2*J1R+T3*J0R)*G(1)
 DMDA(2) =-DMDA(1)
 DKDA(1)=DKDA(1)*(T1*Y1RP+T2*Y1R+T3*Y0R)*G(1)
 DKDA(2)= -DKDA(1)
 T1 =BETA0*GAMMA*K*OMEGA
 T2 = T1*MU0*CAPB
 T1 = T1*EPS0*CAPA
 X= GAMMA*A
 CALL HANKEL(1,X,HF,DUM,F0,DUM)
 TERM = GAMMA*J0R-RHO*F0/HF(1) *J1R
 TERM =A*TERM/(GAMMA*GAMMA + RHO*RHO)
 TERM = TERM-J1R/RHO/GAMMA
 T1 = T1*TERM   $ T2 = T2*TERM
 T1 = T1 + KSQ*CAPB*J1R
 T2 = T2 +KSQ*CAPA*J1R
 TERM =GAMMA*Y0R-RHO*F0/HF(1)*Y1R
 TERM = TERM*A/(GAMMA*GAMMA+KSQ) -Y1R/GAMMA/K
 T3 =BETA0*GAMMA*K*OMEGA
 T4 = T3*MU0*CAPB*TERM
  T3 =T3*EPS0*CAPA*TERM
 T3 = T3 +KSQ*Y1R*CAPB
 T4 = T4 +KSQ*Y1R*CAPA
```

```
      DC 10 I=1,2
   10 IRZ(I) =      (T1*DHDA(I) +T2*DKDA(I) +T3*DIDA(I)+T4*DMDA(I) )
      IRZ1 = IRZ(1) $ IRZ2 = IRZ(2)
      RETURN $ END
      SUBROUTINE FOGC
      COMMON  A,         BETA,      BETA0,     BINT,      CAPA,      CAPB,
     2DHDA(2),DIDA(2),DKDA(2),  DMDA(2), EPS0 ,    H(2),      FOG(2),
     3FUNC,    Q(2),      GAMMA,     IRZ1,      IRZ2,      J0R,       J0S,
     4 J1R,    J1RP,      J1S,       J1SP,      K,         KAPPA,     KSQ,
     5LENGTH, MU,         MU0,       NSQ,       OMEGA,     P,         PI,
     6 RHO,    SIGMA,     Y0R,       Y0S,       Y1R,       Y1RP,      Y1S,
     7 Y1SP
      REAL J0S,J1S,J1SP,J0R,J1R,J1RP,NSQ,K,MU0
      REAL IRZ1,IRZ2,KAPPA,KSQ,LENGTH,MU
      CALL BESSIN
      B=RHO*J1SP*Y1R/SIGMA
      C=(NSQ-1.)*K*BETA*J1S/(A*RHO*SIGMA*SIGMA)
      F=C*J1R $ C=C*Y1R
      D=RHO*J1SP*J1R $E=J1S*J1RP $ G= J1S*Y1RP
      D=D/SIGMA
      FOG=(G-B)*(G-B)+(E-D)*(E-D)+C*C+F*F
      FOG=FOG/((G-NSQ*B)*(G-NSQ*B)+(E-NSQ*D)*(E-NSQ*D)+C*C+F*F)
      FOG = SQRT(MU0*FOG/EPS0)
C     POWER CALCULATION
      DC 5 I=1,2
      FAC=SQRT(MU0/EPS0)/FOG(I)
      P=(G-NSQ*B+C*FAC)**2+(E-NSQ*D+F*FAC)**2
      P=P+(C+(G-B)*FAC)**2 +(F+(E-D)*FAC)**2
      P=.125*PI*PI*PI*A*A*BETA*OMEGA*EPS0*P/RHO
      IF(P.LT.0.) P=-P
      H(I)=SQRT(1./P) $ Q(I) =H(I)/FOG(I)
    5 FOG(2)=-FOG(1)
      P=1.
      RETURN $ END
      SUBROUTINE DHDAC
      COMMON  A,         BETA,      BETA0,     BINT,      CAPA,      CAPB,
     2DHDA(2),DIDA(2),DKDA(2),  DMDA(2), EPS0 ,    F(2),      FOG(2),
     3FUNC,    G(2),      GAMMA,     IRZ1,      IRZ2,      J0R,       J0S,
     4 J1R,    J1RP,      J1S,       J1SP,      K,         KAPPA,     KSQ,
     5LENGTH, MU,         MU0,       NSQ,       OMEGA,     P,         PI,
     6 RHO,    SIGMA,     Y0R,       Y0S,       Y1R,       Y1RP,      Y1S,
     7 Y1SP
      REAL IRZ1,IRZ2,KAPPA,KSQ,LENGTH,MU
      REAL J0S,J1S,J1SP,J0R,J1R,J1RP,NSQ,K,MU0
      TERM1 =    SIGMA*SIGMA-NSQ*RHO*RHO
      TERM1= TERM1*A*J0S*Y1RP/SIGMA
      S= SIGMA*W $ S=1.-2./S/S
      R=RHO*A $S=2./R-R+NSQ*R*S
      S=S*J1S*Y1R $ TERM1=TERM1+S+(NSQ*RHO*RHO/SIGMA/SIGMA-1.)*J1S*Y0R
      S= OMEGA*EPS0*RHO*SIGMA*SIGMA
      S= (NSQ-1.)*K*K*BETA/S
      S=S*(SIGMA*J0S*Y1R+RHO*J1S*Y0R-2.*J1S*Y1R/A)
      DC 5 I=1,2
    5 DHDA(I)=.5*PI*RHO*(TERM1*F(I) +S*G(I) )
      RETURN $ END
      SUBROUTINE DICAC
      COMMON  A,         BETA,      BETA0,     BINT,      CAPA,      CAPB,
     2DHDA(2),DIDA(2),DKDA(2),  DMDA(2), EPS0 ,    F(2),      FOG(2),
     3FUNC,    G(2),      GAMMA,     IRZ1,      IRZ2,      J0R,       J0S,
     4 J1R,    J1RP,      J1S,       J1SP,      K,         KAPPA,     KSQ,
     5LENGTH, MU,         MU0,       NSQ,       OMEGA,     P,         PI,
     6 RHO,    SIGMA,     Y0R,       Y0S,       Y1R,       Y1RP,      Y1S,
     7 Y1SP
      REAL IRZ1,IRZ2,KAPPA,KSQ,LENGTH,MU
```

```
  REAL J0S,J1S,J1SP,J0R,J1R,J1RP,NSQ,K,MU0
  TERM5=SIGMA*J0S*J1R+RHO*J1S*J0R-2.*J1S*J1R/A
  TERM5=(NSQ-1.)*K*K*BETA/(OMEGA*EPS0*RHO*SIGMA*SIGMA)*TERM5
  TERM1 = SIGMA*SIGMA-NSQ*RHO*RHO
  TERM1=TERM1*A*J0S*J1RP/SIGMA
  TERM2=2./RHO/A-RHO*A+NSQ*(RHO*A-2.*RHO/SIGMA/SIGMA/A)
  TERM2 = TERM2*J1S*J1R$ TERM3=NSQ*RHO*RHO/SIGMA/SIGMA-1.
  TERM3 = TERM3*J1S*J0R
  TERM1=  TERM1 +TERM2 + TERM3
   DO 5 I=1,2
5 DIDA(I) =-.5*PI*RHO*(TERM1*F(I) +TERM5*G(I) )
  RETURN $ END
  SUBROUTINE DKDAC
  COMMON  A,         BETA,     BETA0,    BINT,     CAPA,     CAPB,
 2DHDA(2),DIDA(2),DKDA(2), DMDA(2), EPS0 ,   F(2),     FOG(2),
 3FUNC,    G(2),     GAMMA,    IRZ1,     IRZ2,     J0R,      J0S,
 4 J1R,    J1RP,     J1S,      J1SP,     K,        KAPPA,    KSC,
 5LENGTH, MU,        MU0,      NSQ,      OMEGA,    P,        PI,
 6 RHO,    SIGMA,    Y0R,      Y0S,      Y1R,      Y1RP,     Y1S,
 7 Y1SP
  REAL MU
  REAL IRZ1,IRZ2,KAPPA,KSC,LENGTH
  REAL J0S,J1S,J1SP,J0R,J1R,J1RP,NSQ,K,MU0
  TERM1=OMEGA*MU*SIGMA*RHO
  TERM1=BETA*(J0S*Y1R*SIGMA+ RHO*J1S*Y0R-2.*J1S*Y1R/A)/TERM1
  TERM2=(A*J0S*Y1RP+2.*J1S*Y1R/RHO/SIGMA/A-J1S*Y0R/SIGMA)
  DO 5 I=1,2
5 DKDA(I)=
 =      PI*RHO*.5*(NSQ-1.)*K*K/SIGMA*(TERM1*F(I) + TERM2*G(I) )
  RETURN $ END
  SUBROUTINE DMDAC
  COMMON  A,         BETA,     BETA0,    BINT,     CAPA,     CAPB,
 2DHDA(2),DIDA(2),DKDA(2), DMDA(2), EPS0 ,   F(2),     FOG(2),
 3FUNC,    G(2),     GAMMA,    IRZ1,     IRZ2,     J0R,      J0S,
 4 J1R,    J1RP,     J1S,      J1SP,     K,        KAPPA,    KSC,
 5LENGTH, MU,        MU0,      NSQ,      OMEGA,    P,        PI,
 6 RHO,    SIGMA,    Y0R,      Y0S,      Y1R,      Y1RP,     Y1S,
 7 Y1SP
  REAL MU
  REAL J0S,J1S,J1SP,J0R,J1R,J1RP,NSQ,K,MU0
  REAL KAPPA,KSC,LENGTH
  S= OMEGA*MU*SIGMA*RHO
  TERM1=BETA/S*(SIGMA*J0S*J1R+RHO*J1S*J0R-2.*J1S*J1R/A)
  TERM =A*J0S*J1RP+2.*J1S*J1R/RHO/SIGMA/A
  TERM = TERM -J1S*J0R/SIGMA
  DO 5 I=1,2
5 DMDA(I)=
 =      -.5*PI*RHO*(NSQ-1.)*K*K/SIGMA*(TERM*G(I) + TERM1*F(I) )
  RETURN $ END
  SUBROUTINE ERROR
  COMMON ZZ(49)
  REWIND 6
  WRITE(6,5) (ZZ(I),I=1,49)
5 FORMAT(10E12.5)
  RETURN $ END
  SUBROUTINE GAMSCLV
  COMMON  A,         BETA,     BETA0,    BINT,     CAPA,     CAPB,
 2DHDA(2),DIDA(2),DKDA(2), DMDA(2), EPS0 ,   F(2),     FOG(2),
 3FUNC,    G(2),     GAMMA,    IRZ1,     IRZ2,     J0R,      J0S,
 4 J1R,    J1RP,     J1S,      J1SP,     K,        KAPPA,    KSC,
 5LENGTH, MU,        MU0,      NSQ,      OMEGA,    P,        PI,
 6 RHO,    SIGMA,    Y0R,      Y0S,      Y1R,      Y1RP,     Y1S,
 7 Y1SP
  COMMON /NUMBER/ KMAX,NMAX,IPT,IPTM1,AE,ALEN,SWITCH,Z,DZ
```

```
      COMMON /ARY/ AARRAY
      COMMON /BRY/ BARRAY
      COMMON /GRY/ GARRAY
      DIMENSION GARRAY(1),AARRAY(1),BARRAY(1)
      REAL J0,J1
      REAL LOA,NU,MU
      REAL LENGTH
      REAL K,KAPPA,KSQ,MU0,NSQ
      DIMENSION HF(3)
      Z=0 $ DZ=LENGTH/FLOAT(NMAX) $NMAX1=NMAX+1
      DO 5 II=1,NMAX1
      CALL ITERATE
      IF(GAMMA .LT.0.) RETURN
    4 BETA0 = KSQ+ GAMMA*GAMMA
      KAPPA = SQRT(NSQ*KSQ-BETA0)
      BETA0 = SQRT(BETA0)
      X= KAPPA*A $ J0=BEJ0(X) $J1 = BEJ1(X)
      T1= NSQ*(J0/J1-1./X)/X
      X=GAMMA*A $ CALL HANKEL(1,X,HF,DUM,F0,DUM)
      F1= SQRT(EPS0/MU0)
      T2=(-F0/HF(1)-1./X)/X
      FAC = (KAPPA*KAPPA + GAMMA * GAMMA)*BETA0*A
      FAC = -K*A*KAPPA*A*KAPPA*A*F1/FAC *GAMMA*GAMMA
      BOA = FAC*(T1+T2) $BOASQ = BOA*BOA
      T1 = KAPPA*A*KAPPA*A *(J0*J0+J1*J1) -2.*J1*J1
      T1 = T1*(NSQ+ MU0*BOASQ/EPS0)*K*BETA0/KAPPA/KAPPA/KAPPA/KAPPA
      T2= X*X*(1.-F0/HF(1) )+2.
      T2= K*BETA0*T2/GAMMA/GAMMA/GAMMA/GAMMA *J1*J1
      T2= T2*(1.+MU0*BOASQ/EPS0)
      T3 =(BETA0*BETA0+NSQ*KSQ)/KAPPA/KAPPA/KAPPA/KAPPA
      T3 =T3-(BETA0*BETA0+KSQ)/GAMMA/GAMMA/GAMMA/GAMMA
      T3 = T3*BOA*2.*SQRT(MU0/EPS0)*J1*J1
      P=.25*PI*(T1+T2+T3)*SQRT(EPS0/MU0)
      IF(P.LT.0.) P=-P
      CAPA = SQRT(1./P) $ P=1.
      CAPB = CAPA*BOA
      GARRAY(II)=GAMMA $AARRAY(II)=CAPA $ BARRAY(II)=CAPB
      Z=Z+DZ
    5 A=FN(Z)
      NN=0 $ RETURN $ ENTRY GAMFIND
      NN = NN+1  $ GAMMA = GARRAY(NN) $ CAPA = AARRAY(NN) $ CAPB =BARRAY(NN)
     Y(NN)
       BETA0=KSQ+GAMMA*GAMMA $KAPPA=SQRT(NSQ*KSQ-BETA0)
      BETA0= SQRT(BETA0)
       IF(NN.GE.NMAX1)  NN=    0
      RETURN $ END
      SUBROUTINE ITERATE
      COMMON  A,         BETA,     BETA0,    BINT,     CAPA,     CAPB,
     2DHDA(2),DIDA(2),DKDA(2),  DMDA(2), EPS0 ,   F(2),     FOG(2),
     3FUNC,   G(2),     GAMMA,    IRZ1,     IRZ2,     J0R,      J0S,
     4 J1R,   J1RP,     J1S,      J1SP,     K,        KAPPA,    KSQ,
     5LENGTH, MU,       MU0,      NSQ,      OMEGA,    P,        PI,
     6 RHO,   SIGMA,    Y0R,      Y0S,      Y1R,      Y1RP,     Y1S,
     7 Y1SP
      COMMON /NUMBER/ KMAX,NMAX,IPT,IPTM1,AE,ALEN,SWITCH,Z,DZ
      REAL   K,KSQ,  NSQ,MU0,NSQP1,KAPPA
      DIMENSION 9(200)
      INDEX=0
      JJMAX=0
      KAPPA=K*SQRT(          NSQ-1.)
      X=KAPPA*A$    X=(BEJ0 (X)/BEJ1 (X)        )/X
      NSQP1=NSQ+1 $ X=.5*NSQP1*    X
      GAMMA = 1.123*EXP(-X)/A    $ DGAMMA = .1*GAMMA
      QN=Q(A,GAMMA)  $ AMIN =ABS(QN*1.E-6)
```

```
 2 DC 12 I=1,60
   QL =QN
   GAMMA = GAMMA + DGAMMA $ QN= Q(A,GAMMA)
   JJ=2*I  $JJP=2*I-1 $B(JJ)=GAMMA $ B(JJP)=QN $ JJMAX =JJMAX+2
10 IF(QL*QN) 3,3,12
12 CONTINUE
   GC TO 6
 3 GAMMA = GAMMA - DGAMMA
 7 EST=GAMMA + DGAMMA*QL/(QL-QN)
   QE =Q(A,EST) $ IF(ABS(QB).GT.AMIN) GO TO 8
   GAMMA = EST $ RETURN
 8 INDEX= INDEX +1 $ IF(INDEX.GT.70) GO TO 6
   IF(QB*QL)1,1,5
 1 DGAMMA = EST-GAMMA $ QN = QB $ GO TO 7
 5 DGAMMA = GAMMA +DGAMMA-EST$ GAMMA =EST $QL=QB$GO TO 7
 6 PRINT 4,INDEX,GAMMA,QN,DGAMMA,QL
 4 FORMAT(1H0,*THE ITERATION DID NOT CONVERGE*,/,I5,4E15.9)
   PRINT 88,A
   PRINT 88,( B(JJ),JJ=1,JJMAX)
88 FORMAT(2E12.5)
   GAMMA = -1.
   RETURN $ END
   FUNCTION Q(A,GAMMA)
   COMMON  ZZ,       BETA,     BETA0,    BINT,     CAPA,     CAPE,
  2DHDA(2),DIDA(2),DKDA(2),  DMDA(2), EPS0 ,    F(2),     FOG(2),
  3FUNC,    G(2),     GOMMA,    IRZ1,     IRZ2,     J0R,      J0S,
  4 J1R,    J1RP,     J1S,      J1SP,     K,        KAPPA,    KSC,
  5LENGTH, MU,        MU0,      NSQ,      OMEGA,    P,        PI,
  6 RHO,   SIGMA,     Y0R,      Y0S,      Y1P,      Y1RP,     Y1S,
  7 Y1SP
   REAL J0,J1,NSQ,K,KAP,KSQ,KAPSQ
   DIMENSION HF(3).
   BETSQ=KSQ+GAMMA*GAMMA  $BETA0=SQRT(BETSQ)  $ GAMSQ=GAMMA*GAMMA
   KAPSQ = NSQ*KSQ-BETSQ
   KAP=SQRT(KAPSQ)
   X= KAP*A $ J0=BEJ0(X) $ J1= BEJ1(X)
   TERM1 =(J0/J1-1./X)*A*GAMSQ/KAP
   X = GAMMA*A
   CALL HANKEL(1,X,HF,DUM,F0,DUM)
   TERM2 =-X*F0/HF(1)
   FAC=(NSQ*TERM1+TERM2)*(TERM1+TERM2)-(NSQ*TERM1+TERM2)-(TERM1+TERM2
  2)
   TERM1=((NSQ*NSQ-1.)*KSQ-GAMSQ)*GAMSQ/KAPSQ/KAPSQ
   Q=FAC-TERM1
    RETURN    $ END
   FUNCTION FN(Z)
   COMMON /NUMBER/ KMAX,NMAX,IPT,IPTM1,AE,ALEN,SWITCH
   IF(SWITCH) 2,2,4
 2 FN = D4 +C1*EXP(D2*Z) $ RETURN
 4 FN = D1 +C2*Z $ RETURN
   ENTRY FNP
   IF(SWITCH) 3,3,5
 3 FN = D3*EXP(D2*Z) $ RETURN
 5 FN = D2 $ RETURN
   ENTRY INIT
   IF(SWITCH) 7,7,6
 7 D1 = (Z-AE)/(1.-EXP(SWITCH) ) $ D2 = SWITCH/ALEN $ D3=D1*D2
   D4 = Z -D1           $ PRINT 1,D2    $ RETURN
 1 FORMAT(*0EXPONENTIAL TAPER A(Z) = D4 + D1 EXP(*,E10.4,*Z)* )
 6 PRINT 8
 8 FORMAT(*0LINEAR TAPER (MARCUSE) A = A1 + (A2-A1)Z*  )
   D1 = Z $ D2 = (AE-Z)/ALEN
   RETURN $ END
```

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